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Chapter 6
INDUCTION AND FRACTIONAL HORSEPOWER MOTORS

6.1 INTRODUCTION

In this chapter, we will discuss three-phase induction motors and their performance characteristics. We will then discuss motors of the fractional-horsepower class used for applications requiring low power output, small size, and reliability. Standard ratings for this class range from \( \frac{1}{20} \) to 1 hp. Motors rated for less than \( \frac{1}{20} \) hp are called subfractional-horsepower motors and are rated in millihorsepower and range from 1 to 35 mhp. These small motors provide power for all types of equipment in the home, office, and commercial installations. The majority are of the induction-motor type and operate from a single-phase supply.

6.2 THREE-PHASE INDUCTION MOTORS

The induction motor is characterized by simplicity, reliability, and low cost, combined with reasonable overload capacity, minimal service requirements, and good efficiency. An induction motor utilizes alternating current supplied to the stator directly. The rotor receives power by induction effects. The stator windings of an induction motor are similar to those of the synchronous machine. The rotor may be one of two types. In the wound rotor motor, windings similar to those of the stator are employed with terminals connected to insulated slip rings mounted on the shaft. The rotor terminals are made available through carbon brushes bearing on the slip rings. The second type is called the squirrel-cage rotor, where the windings are simply conducting bars embedded in the rotor and short-circuited at each end by conducting end rings.

When the stator of the motor is supplied by a balanced three-phase alternating current source, it will produce a magnetic field that rotates at synchronous speed as determined by the number of poles and applied frequency \( f_s \).

\[
n_s = \frac{120 f_s}{P} \text{ r/min}
\]  

(6.1)

In steady state, the rotor runs at a steady speed \( n_r \) r/min in the same direction as the rotating stator field. The speed \( n_r \) is very close to \( n_s \) when the motor is running low, and is lower as the mechanical load is increased. The speed difference \( n_s - n_r \) is termed the slip and is commonly defined as a per unit value \( s \).
\[ s = \frac{n_s - n_r}{n_s} \]  \hspace{1cm} (6.2)\]

Because of the relative motion between stator and rotor, induced voltages will appear in the rotor with a frequency \( f_r \), called the slip frequency:

\[ f_r = sf_s \]  \hspace{1cm} (6.3)\]

From the above we observe that the induction motor is simply a transformer but that it has a secondary frequency \( f_r \).

**Example 6.1**

Determine the number of poles, the slip, and the frequency of the rotor currents at rated load for three-phase, induction motors rated at:

A. 2200 V, 60 Hz, 588 r/min.
B. 120 V, 600 Hz, 873 r/min.

**Solution**

We use \( P = \frac{120f}{n_r} \), to obtain \( P \), using \( n_r \), the rotor speed given to obtain the slip.

A.

\[ P = \frac{120 \times 60}{588} = 12.245 \]

But \( P \) should be an even number. Therefore, take \( P = 12 \). Hence

\[ n_s = \frac{120f}{P} = \frac{120 \times 60}{12} = 600 \text{ r/min} \]

The slip is thus given by

\[ s = \frac{n_s - n_r}{n_s} = \frac{600 - 588}{600} = 0.02 \]

The rotor frequency is

\[ f_r = sf_s = 0.02 \times 60 = 1.2 \text{ Hz} \]

B.

\[ P = \frac{120 \times 600}{873} = 82.47 \]

Take \( P = 82 \).
Equivalent Circuits

An equivalent circuit of the three-phase induction motor can be developed on the basis of the above considerations and transformer models. Looking into the stator terminals, the applied voltage $V_s$ will supply the resistive drop $I_s R_1$ as well as the inductive voltage $j I_s X_1$ and the counter EMF $E_1$ where $I_s$ is the stator current and $R_1$ and $X_1$ are the stator effective resistance and inductive reactance respectively. In a manner similar to that employed for the analysis of the transformer, we model the magnetizing circuit by the shunt conductance $G_c$ and inductive susceptance $-j B_m$.

The rotor's induced voltage $E_{2s}$ is related to the stator EMF $E_1$ by

$$ E_{2s} = sE_1 $$

(6.4)

This is due simply to the relative motion between stator and rotor. The rotor current $I_r$ is equal to the current $I_s$ in the stator circuit. The induced EMF $E_{2s}$ supplies the resistive voltage component $I_r R_2$ and inductive component $j I_r (sX_2)$. $R_2$ is the rotor resistance, and $X_2$ is the rotor inductive reactance on the basis of the stator frequency.

$$ E_{2s} = I_r R_2 + j I_r (sX_2) $$

or

$$ sE_1 = I_r R_2 + j I_r (sX_2) $$

(6.5)

From the above we conclude that the equivalent rotor impedance seen from the stator is given by:

$$ \frac{E_1}{I_r} = \frac{R_2}{s} + jX_2 $$

The complete equivalent circuit of the induction motor is shown in Figure 6.1.

Considering the active power flow into the induction machine, we find that the input power $P_s$ supplies the stator $\bar{F} R$ losses and the core losses. The remaining power denoted by the air-gap power $P_g$ is that transferred to the rotor circuit. Part of the air-gap power is expended as rotor $\bar{F} R$ losses with the remainder being the mechanical power delivered to the motor shaft. We can express the air-gap power as
The rotor $I^2R$ losses are given by

$$P_{tr} = 3I_r^2R_2$$  \hspace{1cm} (6.7)

As a result, the mechanical power output (neglecting mechanical losses) is

$$P_r = P_g - P_{tr} = 3I_r^2 \left( \frac{1-s}{s} \right) R_2$$  \hspace{1cm} (6.8)

The last formula suggests a splitting of $R_2/s$ into the sum of $R_2$ representing the rotor resistance and a resistance

$$\left( \frac{1-s}{s} \right) R_2$$

which is the equivalent resistance of the mechanical load. As a result, it is customary to modify the equivalent circuit to the form shown in Figure 6.2.

**Motor Torque**

The torque $T$ developed by the motor is related to $P_r$ by

$$T = \frac{P_r}{\omega_r}$$  \hspace{1cm} (6.9)

with $\omega_r$ being the angular speed of the rotor. Thus,
The angular synchronous speed $\omega_s$ is given by

$$\omega_s = \frac{2\pi n_s}{60}$$  \hspace{1cm} (6.11)

As a result, the torque is given by

$$T = \frac{3I_2^2(R_2)}{s\omega_s}$$  \hspace{1cm} (6.12)

The torque is slip-dependent. It is customary to utilize a simplified equivalent circuit for the induction motor in which the shunt branch is moved to the voltage source side. This situation is shown in Figure 6.3. The stator resistance and shunt branch can be neglected in many instances.

**Rotor Current**

On the basis of the approximate equivalent circuit, we can find the rotor
current as

\[ I_r = \frac{V_1}{R_1 + \frac{R_2}{s} + jX_T} \]  \hspace{1cm} (6.13)

At starting, we have \( \omega_s = 0 \); thus \( s = 1 \). The rotor starting current is hence given by

\[ I_{rs} = \frac{V_1}{(R_1 + R_2) + jX_T} \]  \hspace{1cm} (6.14)

The starting current is much higher than the normal (or full-load) current. Depending on the motor type, the starting current can be as high as six to seven times the normal current.

**Example 6.2**
A 15-hp, 220-V, three-phase, 60-Hz, six-pole, Y-connected induction motor has the following parameters per phase:

- \( R_1 = 0.15 \text{ ohm} \)
- \( R_2 = 0.1 \text{ ohm} \)
- \( X_T = 0.5 \text{ ohm} \)
- \( G_c = 6 \times 10^{-3} \)
- \( B_m = 0.15 \text{ S} \)

The rotational losses are equal to the stator hysteresis and eddy-current losses. For a slip of 3 percent, find the following:

A. the line current and power factor;
B. the horsepower output;
C. the starting torque.

**Solution**
A. The voltage specified is line-to-line value as usual. Utilizing the approximate equivalent circuit of Figure 6.3, the rotor current can be seen to be given by

\[ I_r = \frac{220}{\sqrt{3}} \left( \frac{0.15 + 0.1}{0.03} \right) + j0.5 \]

\[ = 36.09 \angle -8.17^\circ \text{ A} \]

The no-load current \( I_\phi \) is obtained as
\[ I_\phi = \frac{220}{\sqrt{3}} \left( 6 \times 10^{-3} - j0.15 \right) \]
\[ = 0.7621 - j19.05 \text{ A} \]

As a result, the line current (stator current) is

\[ I_s = I_c + I_\phi \]
\[ = 43.772 \angle -33.535^\circ \]

Since \( V_1 \) is taken as reference, we conclude that

\[ \phi_s = 33.535^\circ \]
\[ \cos \phi_s = 0.8334 \]

B. The air-gap power is given by

\[ P_g = 3I_s^2 \left( \frac{R_s}{s} \right) = 3(36.09)^2 \left( \frac{0.1}{0.03} \right) = 13,024.881 \text{ W} \]

The mechanical power to the shaft is

\[ P_m = (1 - s)P_g = 12,634.135 \text{ W} \]

The core losses are

\[ P_c = 3E^2 (G_c) = 290.4 \text{ W} \]

The rotational losses are thus

\[ P_{rl} = 290.4 \text{ W} \]

As a result, the net output mechanical power is

\[ P_{out} = P_m - P_{rl} \]
\[ = 12,343.735 \text{ W} \]

Therefore, in terms of horsepower, we get

\[ hP_{out} = \frac{12,343.735}{746} = 16.547 \text{ hp} \]

C. At starting, \( s = 1 \):
\[ |I_t| = \frac{220}{\sqrt{3}} \frac{1}{(0.15 + 0.1) + j0.5} = 227.215 \text{ A} \]

\[ P_g = 3(227.215)^2(0.1) = 15,487.997 \text{ W} \]

\[ \omega_s = \frac{2\pi(60)}{3} = 40\pi \]

\[ T = \frac{P_g}{\omega_s} = \frac{15,487.997}{40\pi} = 123.25 \text{ N.m.} \]

The following script implements Example 6.2 in MATLAB®:

```matlab
% Example 6-2

V=220/3^.5;
s=0.03;
f=60;
R1=0.15;
R2=0.1;
Xt=0.5;
Gc=6*10^-3;
Bm=0.15;
Ir=V/((R1+R2/s)+i*Xt);
abs(Ir)
angle(Ir)*180/pi
Iphi=V*(Gc-i*Bm)
Is=Ir+Iphi;
abs(Is)
angle(Is)*180/pi

% V1 is taken as reference
phi_s=-angle(Is);

pf=cos(phi_s)

% B. The airgap power
Pg=3*(abs(Ir))^2*(R2/s)
% The mechanical power to the shaft
Pm=(1-s)*Pg
% The core loss
Pc=3*E1^2*Gc
% The rotational losses
Prl=Pc
% The net output mechanical power
Pout=Pm-Prl
hpout=Pout/746
```
MATLAB™ con’t.

% At starting s=1
s=1;
Ir=V/((R1+R2/s)+i*Xt);
abs(Ir)
angle(Ir)*180/pi
Pg=3*(abs(Ir))^2*(R2/s)
omega_s=2*pi*f/3;
T=Pg/omega_s

The results obtained from MATLAB™ are as follows:

EDU»
ans =  36.0943
ans =  -8.1685
Iphi =  0.7621-19.0526i
ans =  43.7750
ans =  -33.5313
pf =  0.8336
Pg =  1.3028e+004
Pm =  1.2637e+004
Pc =  290.4000
Prl =  290.4000
Pout =  1.2347e+004
hpout =  16.5506
ans =  227.2150
ans =  -63.4349
Pg =  1.5488
T =  123.2496

6.3 TORQUE RELATIONS

The torque developed by the motor can be derived in terms of the motor parameters and slip using the expressions given before.

\[
T = \frac{3|V|^2}{\omega_s} \frac{R_2}{s} \left( \frac{R_1 + \frac{R_2}{s}}{s} \right)^2 + X_T^2
\]

Neglecting stator resistance, we have
The torque-slip variations are shown in Figure 6.4.

The torque at which maximum torque occurs as

$$T = \frac{3|V|^2 \omega_s}{\omega_s} \frac{R_2}{s} \left( \frac{R_2}{s} \right)^2 + X_F^2$$

The slip at which maximum torque occurs as

$$s_{\text{max}} = \frac{R_2}{X_F}$$

The value of maximum torque is

$$T_{\text{max}} = \frac{3|V|^2}{2\omega_s X_F}$$

The torque-slip variations are shown in Figure 6.4.

**Example 6.3**

The resistance and reactance of a squirrel-cage induction motor rotor at standstill are 0.125 ohm per phase and 0.75 ohm per phase, respectively. Assuming a transformer ratio of unity, from the eight-pole stator having a phase voltage of 120 V at 60 Hz to the rotor secondary, calculate the following:

A. rotor starting current per phase, and  
B. the value of slip producing maximum torque.

**Solution**

A. At starting, $s = 1$:
\[ I_r = \frac{120}{0.125 + j0.75} = 157.823 \angle -80.538 \text{ A} \]

B.

\[ s_{\text{max}} = \frac{R_r}{X_T} = \frac{0.125}{0.75} = 0.1667 \]

The following script implements Example 6.3 in MATLAB™:

```matlab
% Example 6-3
% A squirrel cage induction motor
Rr=0.125; % ohm
XT=0.75; % ohm
V=120; % Volt
f=60; % Hz
% A. Rotor starting current per phase
% At starting s=1
Ir= V/(Rr+j*XT)
abs(Ir)
angle(Ir)*180/pi
% B. The value of slip producing maximum torque
s_maxT=Rr/XT
```

The results obtained from MATLAB™ are as follows:

```plaintext
EDU>
Ir = 2.5946e+001 - 1.5568e+002i
ans = 157.8230
ans = -80.5377
s_maxT = 0.1667
```

**Example 6.4**

The full-load slip of a squirrel-cage induction motor is 0.05, and the starting current is five times the full-load current. Neglecting the stator core and copper losses as well as the rotational losses, obtain:

A. the ratio of starting torque (st) to the full-load torque (fld), and
B. the ratio of maximum (max) to full-load torque and the corresponding slip.

Solution

\[ s_{\text{fld}} = 0.05 \quad \text{and} \quad I_s = 5I_{\text{fld}} \]

\[ \left( \frac{I_s}{I_{\text{fld}}} \right)^2 = \left( \frac{R_2}{0.05} \right)^2 + X_s^2 \left( \frac{R_2^2 + X_s^2}{R_2^2 + X_s^2} \right) = (5)^2 \]

This gives

\[ \frac{R_2}{X_s} = \sqrt{\frac{24}{375}} \approx 0.25 \]

A.

\[ T = \frac{3I_s^2 (R_2)}{s \omega_s} \]

\[ \frac{T_{\text{st}}}{T_{\text{fld}}} = \frac{I_s^2}{I_{\text{fld}}^2} \left( \frac{s_{\text{fld}}}{s_s} \right) = (5)^2 \frac{0.05}{1} = 1.25 \]

B.

\[ s_{\max, r} = \frac{R_2}{X_s} = 0.25 \]

\[ \frac{T_{\max}}{T_{\text{fld}}} = \frac{I_s^2}{I_{\text{fld}}^2} \left( \frac{s_{\text{fld}}}{s_{\max, r}} \right) \]

\[ \approx \left( \frac{s_{\text{fld}}}{s_{\max, r}} \right) \left( \frac{R_2}{s_{\text{fld}}} \right)^2 + X_s^2 \left( \frac{R_2^2 + X_s^2}{2X_s^2} \right) \]

\[ \approx \left( \frac{s_{\text{fld}}}{s_{\max, r}} \right)^2 + 1 \]

\[ \approx \frac{0.05}{0.25} \left[ (5)^2 + 1 \right] \]

Thus,
The following script implements Example 6.4 in MATLAB:

```matlab
% Example 6-4
% A squirrel cage induction motor
sfld=0.05;
sst=1;
% Ist=5*Ifld;
% ratio1=Ist/Ifld=5
ratio1=5;
%
(ratio1)^2=((R2/sfld)^2+(XT)^2)/(R2^2+(XT)^2)
% (R2/XT)^2*((1/sfld)^2-ratio1^2)=ratio1^2-1
% ratio2=R2/XT
f=[((1/sfld)^2-ratio1^2) 0 -(ratio1^2-1)]
ratio2=roots(f);
ratio2=ratio2(1)
% A. T=3*Ir^2*R2/(sfld*ws)
% ratio3=Tst/Tfld
ratio3=ratio1^2*(sfld/sst)
% B.
s_maxT=ratio2
%Tmax/Tfld=(Imax/Ifld)^2*(sfld/s_maxT)
%=(sfld/s_maxT)*((R2/sfld)^2+XT^2)/(2*X T^2)
% (Tmax/Tfld)=(sfld/s_maxT)*((s_maxT/sfld )^2+1)/2
% ratio4=Tmax/Tfld
ratio4=(sfld/s_maxT)*((s_maxT/sfld)^2+1 )/2
```

The results obtained from MATLAB are as follows:

```
EDU»
f =  375     0   -24
ratio2 =  0.2530
ratio3 =  1.2500
s_maxT =  0.2530
ratio4 =  2.6286
```
6.4 CLASSIFICATION OF INDUCTION MOTORS

Integral-horsepower, three-phase, squirrel-cage motors are available from manufacturers’ stock in a range of standard ratings up to 200 hp at standard frequencies, voltages, and speeds. (Larger motors are regarded as special-purposed.) Several standard designs are available to meet various starting and running requirements. Representative torque-speed characteristics of four designs are shown in Figure 6.5. These curves are typical of 1,800 r/min (synchronous-speed) motors in ratings from 7.5 to 200 hp.

The induction motor meets the requirements of substantially constant-speed drives. Many motor applications, however, require several speeds or a continuously adjustable range of speeds. The synchronous speed of an induction motor can be changed by (1) changing the number of poles, (2) varying the rotor resistance, or (3) inserting voltages of the appropriate frequency in the rotor circuits. A discussion of the details of speed control mechanisms is beyond the scope of this work. A common classification of induction motors is as follows.

Class A

Normal starting torque, normal starting current, low slip. This design has a low-resistance, single-cage rotor. It provides good running performance at the expense of starting. The full-load slip is low and the full-load efficiency is high. The maximum torque usually is over 200 percent of full-load torque and occurs at a small slip (less than 20 percent). The starting torque at full voltage

![Figure 6.5 Typical Torque-Speed Curves for 1,800 r/min General-Purpose Induction Motors.](image)
varies from about 200 percent of full-load torque in small motors to about 100 percent in large motors. The high starting current (500 to 800 percent of full-load current when started at rated voltage) is the disadvantage of this design.

**Class B**

Normal starting torque, low starting current, low slip. This design has approximately the same starting torque as the Class A with only 75 percent of the starting current. The full-load slip and efficiency are good (about the same as for the Class A). However, it has a slightly decreased power factor and a lower maximum torque (usually only slightly over 200 percent of full-load torque being obtainable). This is the commonest design in the 7.5 to 200-hp range of sizes used for constant-speed drives where starting-torque requirements are not severe.

**Class C**

High starting torque, low starting current. This design has a higher starting torque with low starting current but somewhat lower running efficiency and higher slip than the Class A and Class B designs.

**Class D**

High starting torque, high slip. This design produces very high starting torque at low starting current and high maximum torque at 50 to 100-percent slip, but runs at a high slip at full load (7 to 11 percent) and consequently has low running efficiency.

### 6.5 ROTATING MAGNETIC FIELDS IN SINGLE-PHASE INDUCTION MOTORS

To understand the operation of common single-phase induction motors, it is necessary to start by discussing two-phase induction machines. In a true two-phase machine two stator windings, labeled $AA'$ and $BB'$, are placed at 90° spatial displacement as shown in Figure 6.6. The voltages $v_A$ and $v_B$ form a set of balanced two-phase voltages with a 90° time (or phase) displacement. Assuming that the two windings are identical, then the resulting flux $\phi_A$ and $\phi_B$ are given by

$$\phi_A = \phi_M \cos \omega t$$

$$\phi_B = \phi_M \cos(\omega t - 90°) = \phi_M \sin \omega t$$

where $\phi_M$ is the peak value of the flux. In Figure 6.6(B), the flux $\phi_A$ is shown to be at right angles to $\phi_B$ in space. It is clear that because of Eqs. (6.17) and (6.18), the phasor relation between $\phi_A$ and $\phi_B$ is shown in Figure 6.6(C) with $\phi_A$...
taken as the reference phasor.

The resultant flux $\phi_P$ at a point $P$ displaced by a spatial angle $\theta$ from the reference is given by

$$\phi_P = \phi_{PA} + \phi_{PB}$$

where $\phi_{PA}$ is the component of $\phi_A$ along the $OP$ axis and $\phi_{PB}$ is the component of $\phi_B$ along the $OP$ axis, as shown in Figure 6.6(D). Here we have

$$\phi_{PA} = \phi_A \cos \theta$$
$$\phi_{PB} = \phi_B \sin \theta$$

As a result, we have

$$\phi_P = \phi_M (\cos \omega t \cos \theta + \sin \omega t \sin \theta)$$

The relationship above can be written alternatively as

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.6.png}
\caption{Rotating magnetic field in a balanced two-phase stator: (A) winding schematic; (B) flux orientation; (C) phasor diagram; and (D) space phasor diagram.}
\end{figure}
The flux at point $P$ is a function of time and the spatial angle $\theta$, and has a constant amplitude $\phi_M$, this result is similar to that obtained earlier for the balanced three-phase induction motor.

The flux $\phi_p$ can be represented by a phasor $\phi_M$ that is coincident with the axis of phase $a$ at $t = 0$. The value of $\phi_p$ is $\phi_M \cos \theta$ at that instant as shown in Figure 6.7(A). At the instant $t = t_1$, the phasor $\phi_M$ has rotated an angle of $\omega t_1$ in the positive direction of $\theta$, as shown in Figure 6.7(B). The value of $\phi_p$ is seen to be $\phi_M \cos (\theta - \omega t_1)$ at that instant. It is thus clear that the flux waveform is a rotating field that travels at an angular velocity $\omega$ in the forward direction of increase in $\theta$.

The result obtained here for a two-phase stator winding set and for a three-phase stator winding set can be extended to an $N$-phase system. In this case the $N$ windings are placed at spatial angles of $\frac{2\pi}{N}$ and excited by sinusoidal voltages of time displacement $\frac{2\pi}{N}$. Our analysis proceeds as follows. The flux waveforms are given by

$$
\phi_1 = \phi_M \cos \omega t \\
\phi_2 = \phi_M \cos \left( \omega t - \frac{2\pi}{N} \right) \\
\vdots \\
\phi_N = \phi_M \cos \left[ \omega t - (i-1) \frac{2\pi}{N} \right]
$$

The resultant flux at a point $P$ can be shown to be given by:

$$
\phi_p = \sum_{i=1}^{N} \phi_{pi}
$$

$$
\phi_p = \frac{N\phi_M}{2} \cos(\theta - \omega t) \quad (6.20)
$$

A rotating magnetic field of constant magnitude will be produced by an $N$-phase winding excited by balanced $N$-phase currents when each phase is displaced $\frac{2\pi}{N}$ electrical degrees from the next phase in space.
In order to understand the operation of a single-phase induction motor, we consider the configuration shown in Figure 6.8. The stator carries a single-phase winding and the rotor is of the squirrel-cage type. This configuration corresponds to a motor that has been brought up to speed, as will be discussed presently.

Let us now consider a single-phase stator winding as shown in Figure 6.9(A). The flux $\phi_a$ is given by

$$\phi_a = \phi_M \cos \omega t$$

(6.21)

The flux at point $P$ displaced by angle $\theta$ from the axis of phase $a$ is clearly given by

$$\phi_P = \phi_a \cos \theta$$

Using Eq. (6.21), we obtain
The flux at point \( P \) can therefore be seen to be the sum of two waveforms \( \phi_f \) and \( \phi_b \) given by

\[
\phi_f = \frac{\phi_M}{2} \cos(\theta - \omega t) \quad \text{(6.23)}
\]

\[
\phi_b = \frac{\phi_M}{2} \cos(\theta + \omega t) \quad \text{(6.24)}
\]

The waveform \( \phi_f \) is of the same form as that obtained in Eq. (6.19), which was shown to be rotating in the forward direction (increase in \( \theta \) from the axis of phase \( a \)). The only difference between Eqs. (6.23) and (6.21) is that the amplitude of \( \phi_f \) is half of that of \( \phi_P \) in Eq. (6.21). The subscript \( f \) in Eq. (6.23) signifies the fact that \( \cos(\theta - \omega t) \) is forward rotating wave.

Consider now the waveform \( \phi_b \) of Eq. (6.24). At \( t = 0 \), the value of \( \phi_b \) is \( \phi_M/2 \) \( \cos \theta \) and is represented by the phasor \( \phi_M/2 \), which is coincident with the axis of phase \( a \) as shown in Figure 6.10(a). Note that at \( t = 0 \), both \( \phi_f \) and \( \phi_b \) are equal in value. At a time instant \( t = t_1 \), the phasor \( \phi_M/2 \) is seen to be at angle \( \alpha t_1 \) with the axis of phase \( a \), as shown in Figure 6.9(B). The waveform \( \phi_b \) can therefore be seen to be rotating at an angular velocity \( \omega \) in a direction opposite to that of \( \phi_f \) and we refer to \( \phi_b \) as a backward-rotating magnetic field. The subscript \( b \) in Eq. (6.24) signifies the fact that \( \cos(\theta + \omega t) \) is a backward-rotating wave.

In a single-phase induction machine there are two magnetic fields rotating in opposite directions. Each field produces an induction-motor torque in a direction opposite to the other. If the rotor is at rest, the forward torque is equal and opposite to the backward torque and the resulting torque is zero.
single-phase induction motor is therefore incapable of producing a torque at rest and is not a self-starting machine. If the rotor is made to rotate by an external means, each of the two fields would produce a torque-speed characteristic similar to a balanced three-phase (or two-phase) induction motor, as shown in Figure 6.11 in the dashed curves. The resultant torque-speed characteristic is shown in a solid line. The foregoing argument will be confirmed once we develop an equivalent circuit for the single-phase induction motor.

6.6 EQUIVALENT CIRCUITS FOR SINGLE-PHASE INDUCTION MOTORS

In a single-phase induction motor, the pulsating flux wave resulting from a single winding stator MMF is equal to the sum of two rotating flux components. The first component is referred to as the forward field and has a constant amplitude equal to half of that of the stator waveform. The forward field rotates at synchronous speed. The second component, referred to as the backward field, is of the same constant amplitude but rotates in the opposite (or backward) direction at synchronous speed. Each component induces its own rotor current and creates induction motor action in the same manner as in a balanced three-phase induction motor. It is on this basis that we conceive of the circuit model of Figure 6.12(A). Note that $R_1$ and $X_1$ are the stator resistance and leakage reactance, respectively, and $V_1$ is the stator input voltage. The EMF $E_1$ is assumed to be the sum of two components, $E_{1_f}$ and $E_{1_b}$, corresponding to the forward and backward field waves, respectively. Note that since the two waves have the same amplitude, we have

$$E_{1_f} = E_{1_p} = \frac{E_1}{2} \quad (6.25)$$

The rotor circuit is modeled as the two blocks shown in Figure 6.12(A) representing the rotor forward circuit model $Z_f$ and the rotor backward circuit model $Z_b$, respectively.

![Figure 6.10](image-url)  
**Figure 6.10** Showing that $\phi_b$ is a backward-rotating wave: (A) $t = 0$; (B) $t = t_s$. 

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The model of the rotor circuit for the forward rotating wave $Z_f$ is simple since we are essentially dealing with induction-motor action and the rotor is set in motion in the same direction as the stator synchronous speed. The model of $Z_f$ is shown in Figure 6.12(B) and is similar to that of the rotor of a balanced three-phase induction motor. The impedances dealt with are half of the actual values to account for the division of $E_1$ into two equal voltages. In this model, $X_m$ is the magnetizing reactance, and $2R'$ and $2X'$ are rotor resistance and leakage reactance, both referring to the stator side. The slip $s_f$ is given by

$$s_f = \frac{n_s - n_r}{n_s}$$  \hspace{1cm} (6.26)

This is the standard definition of slip as the rotor is revolving in the same direction as that of the forward flux wave.

The model of the rotor circuit for the backward-rotating wave $Z_b$ is shown in Figure 6.12(C) and is similar to that of $Z_f$, with the exception of the backward slip, denoted by $s_b$. The backward wave is rotating at a speed of $-n_s$, and the rotor is rotating at $n_r$. We thus have

$$s_b = \frac{(-n_s) - n_r}{-n_s} = 1 + \frac{n_r}{n_s}$$  \hspace{1cm} (6.27)

Using Eq. (6.26), we have

$$s_f = 1 - \frac{n_r}{n_s}$$  \hspace{1cm} (6.28)

As a result, we conclude that the slip of the rotor with respect to the backward
wave is related to its slip with respect to the forward wave by

\[ s_b = 2 - s_f \]  \hspace{1cm} (6.29)

We now let \( s \) be the forward slip,

\[ s_f = s \]  \hspace{1cm} (6.30)

and thus

\[ s_b = 2 - s \]  \hspace{1cm} (6.31)

On the basis of Eqs. (6.30) and (6.31), a complete equivalent circuit as shown in Figure 6.12(D) is now available. The core losses in the present model are treated separately in the same manner as the rotational losses.

![Figure 6.12](image)

Figure 6.12 Developing an equivalent circuit of for single-phase induction motors: (A) basic concept; (B) forward model; (C) backward model; and (D) complete equivalent circuit.
The forward impedance $Z_f$ is obtained as the parallel combination of $(jX_m/2)$ and $[(R'_2/2s) + j(X'_2/2s)]$, given by

$$Z_f = \frac{j(X_m/2)[(R'_2/2s) + j(X'_2/2)]}{(R'_2/2s) + j((X_m + X'_2)/2)} \quad (6.32)$$

Similarly, for the backward impedance, we get

$$Z_b = \frac{j(X_m/2)[(R'_2/2(2-s)) + j(X'_2/2)]}{[R'_2/2(2-s) + j((X_m + X'_2)/2)]} \quad (6.33)$$

Note that with the rotor at rest, $n_r = 0$, and thus with $s = 1$, we get $Z_f = Z_b$.

**Example 6.5**

The following parameters are available for a 60-Hz four-pole single-phase 110-V ½-hp induction motor:

- $R_1 = 1.5 \ \Omega$
- $R'_2 = 3 \ \Omega$
- $X_1 = 2.4 \ \Omega$
- $X'_2 = 2.4 \ \Omega$
- $X_m = 73.4 \ \Omega$

Calculate $Z_f$, $Z_b$, and the input impedance of the motor at a slip of 0.05.

**Solution**

$$Z_f = \frac{j(36.7(30 + j1.2))}{30 + j37.9} = 22.796 \angle 40.654^\circ$$

$$= 17.294 + j14.851 \ \Omega$$

The result above is a direct application of Eq. (6.32). Similarly, using Eq. (6.33), we get

$$Z_b = \frac{j(36.7((1.5/1.95) + j1.2))}{(1.5/1.95) + j37.9} = 1.38 \angle 58.502^\circ \ \Omega$$

$$= 0.721 + j1.766 \ \Omega$$

We observe here that $|Z_f|$ is much larger than $|Z_b|$ at this slip, in contrast to the situation at starting ($s = 1$), for which $Z_f = Z_b$.

The input impedance $Z_i$ is obtained as

$$Z_i = Z_1 + Z_f + Z_b = 19.515 + j18.428$$

$$= 26.841 \angle 43.36^\circ \ \Omega$$

Equations (6.32) and (6.33) yield the forward and backward...
impedances on the basis of complex number arithmetic. The results can be written in the rectangular forms

\[ Z_f = R_f + jX_f \]  \hspace{1cm} (6.34)  

and

\[ Z_b = R_b + jX_b \]  \hspace{1cm} (6.35)  

Using Eq. (6.32), we can write

\[ 2R_f = \frac{a_f X_m^2}{a_f^2 + X_f^2} \]  \hspace{1cm} (6.36)  

and

\[ X_f = \frac{R_f}{a_f X_m} (a_f^2 + X_f X'_2) \]  \hspace{1cm} (6.37)  

where

\[ a_f = \frac{R'_f}{s} \]  \hspace{1cm} (6.38)  

\[ X_f = X'_2 + X_m \]  \hspace{1cm} (6.39)  

In a similar manner we have, using Eq. (6.33),

\[ 2R_b = \frac{a_b X_m^2}{a_b^2 + X_b^2} \]  \hspace{1cm} (6.40)  

\[ X_b = \frac{R_b}{a_b X_m} (a_b^2 + X_b X'_2) \]  \hspace{1cm} (6.41)  

where

\[ a_b = \frac{R'_b}{2 - s} \]  \hspace{1cm} (6.42)  

It is often desirable to introduce some approximations in the formulas just derived. As is the usual case, for \( X_f > 10 a_b \), we can write an approximation to Eq. (6.40) as
As a result, by substitution in Eq. (6.41), we get

\[ X_b \approx \frac{X_m^2}{2X_t} + \frac{a_b R_b}{X_m} \]  \hspace{1cm} (6.44)

We can introduce further simplifications by assuming that \( \frac{X_m}{X_t} \cong 1 \), to obtain from Eq. (6.43)

\[ 2R_b \cong a_b = \frac{R_s^2}{2 - s} \]  \hspace{1cm} (6.45)

Equation (6.44) reduces to the approximate form

\[ X_b \cong X_m^2 \frac{a_b^2}{2X_m} \]  \hspace{1cm} (6.46)

Neglecting the second term in Eq. (6.46), we obtain the most simplified representation of the backward impedance as

\[ R_b = \frac{R_s^2}{2(2 - s)} \]  \hspace{1cm} (6.47)

\[ X_b = \frac{X_m^2}{2} \]  \hspace{1cm} (6.48)

Equations (6.47) and (6.48) imply that \( \frac{X_m}{2} \) is considered an open circuit in the backward field circuit, as shown in Figure 6.13.
6.7 POWER AND TORQUE RELATIONS

The development of an equivalent-circuit model of a running single-phase induction motor enables us to quantify power and torque relations in a simple way. The power input to the stator \( P_i \) is given by

\[
P_i = V_1 I_1 \cos \phi_1
\]

(6.49)

where \( \phi_1 \) is the phase angle between \( V_1 \) and \( I_1 \). Part of this power will be dissipated in stator ohmic losses, \( P_{is} \), given by

\[
P_{is} = I_1^2 R_i
\]

(6.50)

The core losses will be accounted for as a fixed loss and is treated in the same manner as the rotational losses at the end of the analysis. The air-gap power \( P_g \) is thus given by

\[
P_g = P_i - P_{is}
\]

(6.51)

The air-gap power is the power input to the rotor circuit and can be visualized to be made up of two components. The first component is the power taken up by the forward field and is denoted by \( P_{gf} \), and the second is the backward field power denoted by \( P_{gb} \). Thus we have

\[
P_g = P_{gf} + P_{gb}
\]

(6.52)

As we have modeled the forward field circuit by an impedance \( Z_f \) it is natural to write

\[
P_{gf} = |I_1|^2 R_f
\]

(6.53)

Similarly, we write

\[
P_{gb} = |I_1|^2 R_b
\]

(6.54)

The ohmic losses in the rotor circuit are treated in a similar manner. The losses in the rotor circuit due to the forward field \( P_{rf} \) can be written as

\[
P_{rf} = s_f P_{gf}
\]

(6.55)

Similarly, the losses in the rotor circuit due to the backward field are written as

\[
P_{rb} = s_b P_{gb}
\]

(6.56)
Equations (6.55) and (6.56) are based on arguments similar to those used with the balanced three-phase induction motor. Specifically, the total rotor equivalent resistance in the forward circuit is given by

$$R_{rf} = \frac{R'_r}{2s_f}$$

(6.57)

This is written as

$$R_{rf} = \frac{R'_r}{2} + \frac{R'_s(1-s_f)}{2s_f}$$

(6.58)

The first term corresponds to the rotor ohmic loss due to the forward field and the second represents the power to mechanical load and fixed losses. It is clear from Figure 6.14 that

$$P_{rf} = |I_{rf}|^2 \frac{R'_r}{2}$$

(6.59)

and

$$P_{rf} = |I_{rf}|^2 \frac{R'_s}{2s_f}$$

(6.60)

Combining Eqs. (6.59) and (6.60), we get Eq. (6.55). A similar argument leads to Eq. (6.56). It is noted here that Eqs. (6.53) and (6.60) are equivalent, since the active power to the rotor circuit is consumed only in the right-hand branch, with \(jX_m/2\) being a reactive element.

**Figure 6.14** Equivalent circuit of single-phase induction motor showing rotor loss components in the forward and backward circuits.
The net power form the rotor circuit is denoted by $P_m$ and is given by

$$P_m = P_{mf} + P_{mb} \quad (6.61)$$

The component $P_{mf}$ is due to the forward circuit and is given by

$$P_{mf} = P_{gf} - P_{rf} \quad (6.62)$$

Using Eq. (6.55), we get

$$P_{mf} = (1 - s_f)P_{gf} \quad (6.63)$$

Similarly, $P_{mb}$ is due to the backward circuit and is given by

$$P_{mb} = P_{gb} - P_{rb} \quad (6.64)$$

Using Eq. (6.56), we get

$$P_{mb} = (1 - s_b)P_{gb} \quad (6.65)$$

Recall that

$$s_f = s$$
$$s_b = 2 - s$$

As a result,

$$P_{mf} = (1 - s)P_{gf} \quad (6.66)$$
$$P_{mb} = (s - 1)P_{gb} \quad (6.67)$$

We now substitute Eqs. (6.66) and (6.67) into Eq. (6.61), to obtain

$$P_m = (1 - s)(P_{gf} - P_{gb}) \quad (6.68)$$

The shaft power output $P_o$ can now be written as

$$P_o = P_m - P_{rot} - P_{core} \quad (6.69)$$

The rotational losses are denoted by $P_{rot}$ and the core losses are denoted by $P_{core}$.

The output torque $T_o$ is obtained as
If fixed losses are neglected, then

\[ T_m = \frac{P_m}{\omega_s (1-s)} \]  

(6.71)

As a result, using Eq. (6.68), we get

\[ T_m = \frac{1}{\omega_s} (P_{gf} - P_{gb}) \]  

(6.72)

The torque due to the forward field is

\[ T_{mf} = \frac{P_{mf}}{\omega_s} = \frac{P_{gf}}{\omega_s} \]  

(6.73)

The torque due to the backward field is

\[ T_{mb} = \frac{P_{mb}}{\omega_s} = -\frac{P_{gb}}{\omega_s} \]  

(6.74)

It is thus clear that the net mechanical torque is the algebraic sum of a forward torque \( T_{mf} \) (positive) and a backward torque \( T_{mb} \) (negative). Note that at starting, \( s = 1 \) and \( R_f = R_b \), and as a result \( P_{gf} = P_{gb} \), giving zero output torque. This confirms our earlier statements about the need for starting mechanisms for a single-phase induction motor. This is discussed in the next section.

**Example 6.6**

For the single-phase induction motor of Example 6.5, it is necessary to find the power and torque output and the efficiency when running at a slip of 5 percent. Neglect core and rotational losses.

**Solution**

In Example 6.5 we obtained

\[ Z_i = 26.841 \angle 43.36^\circ \]

As a result, with \( V_i = 110 \angle 0 \), we obtain

\[ I_i = \frac{110 \angle 0}{26.841 \angle 43.36^\circ} = 4.098 \angle -43.36^\circ \text{ A} \]
The power factor is thus
\[
\cos \phi_1 = \cos 43.36^\circ = 0.727
\]
The power input is
\[
P_1 = V_1 I_1 \cos \phi_1 = 327.76 \text{ W}
\]
We have from Example 6.5 for \( s = 0.05 \),
\[
R_f = 17.294 \text{ Ω} \quad R_b = 0.721 \text{ Ω}
\]
Thus we have
\[
\begin{align*}
P_{gf} &= |I_1|^2 R_f = (4.098)^2 (17.294) = 290.46 \text{ W} \\
P_{gb} &= |I_1|^2 R_b = (4.098)^2 (0.721) = 12.109 \text{ W}
\end{align*}
\]
The output power is thus obtained as
\[
P_m = (1 - s)(P_{gf} - P_{gb})
= 0.95(290.46 - 12.109) = 264.43 \text{ W}
\]
As we have a four-pole machine, we get
\[
\begin{align*}
n_f &= \frac{120(60)}{4} = 1800 \text{ r/min} \\
\omega_f &= \frac{2\pi n_f}{60} = 188.5 \text{ rad/s}
\end{align*}
\]
The output torque is therefore obtained as
\[
T_m = \frac{1}{\omega_f} (P_{gf} - P_{gb})
= \frac{290.46 - 12.109}{188.5} = 1.4767 \text{ N·m}
\]
The efficiency is now calculated as
\[
\eta = \frac{P_m}{P_1} = \frac{264.43}{327.76} = 0.8068
\]
It is instructive to account for the losses in the motor. Here we have the static ohmic losses obtained as
The forward rotor losses are

\[ P_{f_{sf}} = sP_{gf} = 0.05(290.46) = 14.523 \text{ W} \]

The backward rotor losses are

\[ P_{f_{rb}} = (2 - s)P_{gb} = 1.95(12.109) = 23.613 \text{ W} \]

The sum of the losses is

\[ P_l = 25.193 + 14.523 + 23.613 = 63.329 \text{ W} \]

The power output and losses should match the power input

\[ P_m + P_l = 264.43 + 63.33 = 327.76 \text{ W} \]

which is indeed the case.

Example 6.7

A single-phase induction motor takes an input power of 490 W at a power factor of 0.57 lagging from a 110-V supply when running at a slip of 5 percent. Assume that the rotor resistance and reactance are 1.78Ω and 1.28Ω, respectively, and that the magnetizing reactance is 25Ω. Find the resistance and reactance of the stator.

Solution

The equivalent circuit of the motor yields

\[ Z_j = \frac{\{1.78/2(0.5)\} + j0.64}{j12.5} = 5.6818 + j8.3057 \]

\[ Z_b = \frac{\{1.78/2(1.95)\} + j0.64}{j12.5} = 0.4125 + j0.6232 \]

As a result of the problem specifications

\[ P_i = 490 \text{ W} \quad \cos \phi = 0.57 \quad V = 110 \]

\[ I_i = \frac{P_i}{V \cos \phi} = \frac{590}{110(0.57)} = 7.815 \angle -55.248^\circ \]

Thus the input impedance is
The stator impedance is obtained as

\[ Z_i = Z_r - (Z_f + Z_b) = 1.9287 + j2.636 \, \Omega \]

6.8 **STARTING SINGLE-PHASE INDUCTION MOTORS**

We have shown earlier that a single-phase induction motor with one stator winding is not capable of producing a torque at starting [see, for example, Eq. (6.68) with \( s = 1 \)]. Once the motor is running, it will continue to do so, since the forward field torque dominates that of the backward field component. We have also seen that with two stator windings that are displaced by 90° in space and with two-phase excitation a purely forward rotating field is produced, and this form of a motor (like the balanced three-phase motor) is self-starting.

Methods of starting a single-phase induction motor rely on the fact that given two stator windings displaced by 90° in space, a starting torque will result if the flux in one of the windings lags that of the other by a certain phase angle \( \psi \). To verify this, we consider the situation shown in Figure 6.15. Assume that

\[ \phi_A = \phi_M \cos \omega t \]

\[ \phi_B = \phi_M \cos(\omega t - \psi) \]

Clearly, the flux at \( P \) is given by the sum of \( \phi_{PA} \) and \( \phi_{PB} \)

\[ \phi_{PA} = \frac{\phi_M}{2} [\cos(\theta - \omega t) + \cos(\theta + \omega t)] \]  

\[ \phi_{PB} = \frac{\phi_M}{2} [\cos \psi [\sin(\theta + \omega t) + \sin(\theta - \omega t)] + \sin \psi [\cos(\theta - \omega t) - \cos(\theta + \omega t)]] \]

The flux at \( P \) is therefore obtained as

\[ \phi_P = \frac{\phi_M}{2} \left[ a_f \cos(\theta - \omega t) + a_f \sin(\theta - \omega t) + a_f \cos(\theta + \omega t) + a_f \sin(\theta + \omega t) \right] \]  

where we have
Figure 6.15 Two stator windings to explain the starting mechanism of single-phase induction motors.

\[
\begin{align*}
a_f &= 1 + \sin \psi \\
a_i &= \cos \psi \\
a_b &= 1 - \sin \psi \\
a_i &= \cos \psi
\end{align*}
\]

Note that we can also define the magnitudes \( a_f \) and \( a_b \) by

\[
\begin{align*}
a_f^2 &= a_f^2 + a_f^2 = 2(1 + \sin \psi) \quad (6.80) \\
a_b^2 &= a_b^2 + a_b^2 = 2(1 - \sin \psi) \quad (6.81)
\end{align*}
\]

The angles \( \alpha_f \) and \( \alpha_b \) are defined next by

\[
\begin{align*}
\cos \alpha_f &= \frac{a_f}{a_f} = \sqrt{\frac{1 + \sin \psi}{2}} \\
\cos \alpha_b &= \frac{a_b}{a_b} = \sqrt{\frac{1 - \sin \psi}{2}} \\
\sin \alpha_f &= \frac{a_f}{a_f} = \frac{\cos \psi}{\sqrt{2(1 + \sin \psi)}} \\
\sin \alpha_b &= \frac{a_b}{a_b} = \frac{\cos \psi}{\sqrt{2(1 - \sin \psi)}}
\end{align*}
\]

We can now write the flux \( \phi_p \) as

\[
\phi_p = \frac{\phi_M}{2}[a_f \cos(\theta - \omega t + \alpha_f) + a_b \cos(\theta + \omega t - \alpha_b)] \quad (6.82)
\]
It is clear that $\phi_P$ is the sum of a forward rotating component $\phi_f$ and a backward rotating component $\phi_b$ given by

$$\phi_P = \phi_f(t) + \phi_b(t) \quad (6.83)$$

where

$$\phi_f(t) = \frac{a_f \dot{\phi}_M}{2} \cos(\theta - \omega t + \alpha_f) \quad (6.84)$$

$$\phi_b(t) = \frac{a_b \dot{\phi}_M}{2} \cos(\theta + \omega t - \alpha_b) \quad (6.85)$$

Let us note here that from Eqs. (6.80) and (6.81), we can see that

$$a_f > a_b \quad (6.86)$$

As a result, the magnitude of the forward rotating wave is larger than that of the backward rotating wave. It is clear that for the arrangement of Figure 6.15, a starting torque should result. This is the basis of the starting mechanisms for single-phase induction motors.

### 6.9 SINGLE-PHASE INDUCTION MOTOR TYPES

Single-phase induction motors are referred to by names that describe the method of starting. A number of types of single-phase induction motors are now discussed.

#### Split-Phase Motors

A single-phase induction motor with two distinct windings on the stator that are displaced in space by 90 electrical degrees is called a split-phase motor. The main (or running) winding has a lower $R/X$ ratio than the auxiliary (or starting) winding. A starting switch disconnects the auxiliary windings when the motor is running at approximately 75 to 80 percent of synchronous speed. The switch is centrifugally operated. The rotor of a split-phase motor is of the squirrel-cage type. At starting, the two windings are connected in parallel across the line as shown in Figure 6.16.

The split-phase design is one of the oldest single-phase motors and is most widely used in the ratings of 0.05 to 0.33 hp. A split-phase motor is used in machine tools, washing machines, oil burners, and blowers, to name just a few of its applications.

The torque-speed characteristic of a typical split-phase induction motor
is shown in Figure 6.17. At starting the torque is about 150 percent of its full-load value. As the motor speed picks up, the torque is increased (except for a slight decrease at low speed) and may reach higher than 2505 of full-load value. The switch is opened and the motor runs on its main winding alone and the motor reaches its equilibrium speed when the torque developed is matched by the load.

**Capacitor-Start Motors**

The class of single-phase induction motors in which the auxiliary winding is connected in series with a capacitor is referred to as that of capacitor motors. The auxiliary winding is placed 90 electrical degrees form the main winding. There are three distinct types of capacitor motors in common practice. The first type, which we discuss presently, employs the auxiliary winding and capacitor only during starting and is thus called a capacitor-start motor. It is thus clear that a centrifugal switch that opens at 75 to 80 percent of synchronous speed is used in the auxiliary winding circuit (sometimes called the capacitor phase). A sketch of the capacitor-start motor connection is shown in Figure 6.18. A commercial capacitor-start motor is not simply a split-phase motor with
a capacitor inserted in the auxiliary circuit but is a specially designed motor that produces higher torque than the corresponding split-phase version.

Capacitor-start motors are extremely popular and are available in all ratings from 0.125 hp up. For ratings at 1/3 hp and above, capacitor-start motors are wound as dual-voltage so that they can be operated on either a 115- or a 230-V supply. In this case, the main winding is made of two sections that are connected in series for 230-V operation or in parallel for 115-V operation. The auxiliary winding in a dual-voltage motor is made of one section which is connected in parallel with one section of the main winding for 230-V operation. The auxiliary winding in a dual-voltage motor is made of one section which is connected in parallel with one section of the main winding for 230-V operation.

It is important to realize that the capacitor voltage increases rapidly above the switch-open speed and the capacitor can be damaged if the centrifugal switch fails to open at the designed speed. It is also important that switches not flutter, as this causes a dangerous rise in the voltage across the capacitor.

A typical torque-speed characteristic for a capacitor-start single-phase induction motor is shown in Figure 6.19. The starting torque is very high, which is a desirable feature of this type of motor.

**Permanent-Split Capacitor Motors**

The second type of capacitor motors is referred to as the permanent-split capacitor motor, where the auxiliary winding and the capacitor are retained at normal running speed. This motor is used for special-purpose applications requiring high torque and is available in ratings from $10^3$ to $1/3$ hp. A schematic of the permanent-split capacitor motor is shown in Figure 6.20.

A typical torque-speed characteristic for a permanent-split capacitor motor is shown in Figure 6.21. The starting torque is noticeably low since the capacitance is a compromise between best running and starting conditions. The next type of motor overcomes this difficulty.

![Figure 6.18 Capacitor-start motor.](image)
Figure 6.19 Torque-speed characteristic of a capacitor-start motor.

Figure 6.20 Permanent-split capacitor motor.

Figure 6.21 Torque-speed characteristic of a permanent-split single-phase induction motor.
Two-Value Capacitor Motors

A two-value capacitor motor starts with one value of capacitors in series with the auxiliary winding and runs with a different capacitance value. This change can be done either using two separate capacitors or through the use of an autotransformer. This motor has been replaced by the capacitor-start motor for applications such as refrigerators and compressors.

For the motor using an autotransformer, a transfer switch is used to change the tap on the autotransformer, as shown in Figure 6.22(A). This arrangement appears to be obsolete now and the two-capacitor mechanism illustrated in Figure 6.22(B) is used.

A typical torque-speed characteristic for a two-value capacitor motor is shown in Figure 6.23. Note that optimum starting and running conditions can be accomplished in this type of motor.

Repulsion-Type Motors

A repulsion motor is a single-phase motor with power connected to the stator winding and a rotor whose winding is connected to a commutator. The brushes on the commutator are short-circuited and are positioned such that there is an angle of 20 to 30° between the magnetic axis of the stator winding and the magnetic axis of the rotor winding. A representative torque-speed characteristic for a repulsion motor is shown in Figure 6.24. A repulsion motor is a variable-speed motor.

If in addition to the repulsion winding, a squirrel-cage type of winding is embedded in the rotor, we have a repulsion-induction motor. The torque-speed characteristic for a repulsion-induction motor is shown in Figure 6.25 and can be thought of as a combination of the characteristics of a single-phase induction motor and that of a straight repulsion motor.

A repulsion-start induction motor is a single-phase motor with the same windings as a repulsion motor, but at a certain speed the rotor winding is short circuited to give the equivalent of a squirrel-cage winding. The repulsion-start motor is the first type of single-phase motors that gained wide acceptance. In recent years, however, it has been replaced by capacitor-type motors. A typical torque-speed characteristic of a repulsion-start induction motor is shown in Figure 6.26.

Shaded-Pole Induction Motors

For applications requiring low power of ¼ hp or less, a shaded-pole induction motor is the standard general-purpose device for constant-speed applications. The torque characteristics of a shaded-pole motor are similar to those of a permanent-split capacitor motor as shown in Figure 6.27.
Figure 6.22 Two-value capacitor motor: (A) autotransformer type; (B) two-capacitor type.

Figure 6.23 Torque-speed characteristic of a two-value capacitor motor.
Figure 6.24 Torque-speed characteristic of a repulsion motor.

Figure 6.25 Torque-speed characteristic of a repulsion-induction motor.

Figure 6.26 Torque-speed characteristic of a repulsion-start single-phase induction motor.
PROBLEMS

Problem 6.1
Determine the number of poles, the slip, and the frequency of the rotor currents at rated load for three-phase, induction motors rated at:

A. 220 V, 50 Hz, 1440 r/min.
B. 120 V, 400 Hz, 3800 r/min.

Problem 6.2
A 50-HP, 440-V, three-phase, 60-Hz, six-pole, Y-connected induction motor has the following parameters per phase:

\[
\begin{align*}
R_2 &= 0.15 \text{ ohm} \\
R_1 &= 0.12 \text{ ohm} \\
G_c &= 6 \times 10^{-3} \text{ siemens} \\
X_T &= 0.75 \text{ ohm} \\
B_m &= 0.07 \text{ siemens}
\end{align*}
\]

The rotational losses are equal to the stator hysteresis and eddy-current losses. For a slip of 4 percent, find the following

A. the line current and power factor.
B. the horsepower output.
C. the starting torque.

Problem 6.3
Use MATLAB™ to verify the results of Problem 6.2.
Problem 6.4
The rotor resistance and reactance of a squirrel-cage induction motor rotor at standstill are 0.14 ohm per phase and 0.8 ohm per phase respectively. Assuming a transformer ratio of unity, from the eight-pole stator having a phase voltage of 254 at 60 Hz to the rotor secondary, calculate the following

A. rotor starting current per phase
B. the value of slip producing maximum torque.

Problem 6.5
The full-load slip of a squirrel-cage induction motor is 0.06, and the starting current is five times the full-load current. Neglecting the stator core and copper losses as well as the rotational losses, obtain:

A. the ratio of starting torque to the full-load torque.
B. the ratio of maximum to full-load torque and the corresponding slip.

Problem 6.6
The rotor resistance and reactance of a wound-rotor induction motor at standstill are 0.14 ohm per phase and 0.8 ohm per phase, respectively. Assuming a transformer ratio of unity, from the eight-pole stator having a phase voltage of 254 V at 60 Hz to the rotor secondary, find the additional rotor resistance required to produce maximum torque at:

A. Starting \( s = 1 \)
B. A speed of 450 r/min.

Problem 6.7
A two-pole 60-Hz induction motor develops a maximum torque of twice the full-load torque. The starting torque is equal to the full load torque. Determine the full load speed.

Problem 6.8
The starting torque of a three-phase induction motor is 165 percent and its maximum torque is 215 percent of full-load torque. Determine the slips at full load and at maximum torque. Find the rotor current at starting in per unit of full-load rotor current.

Problem 6.9
Consider a 25-hp, 230-V three-phase, 60-Hz squirrel cage induction motor operating at rated voltage and frequency. The rotor \( \dot{I}R \) loss at maximum torque is 9.0 times that at full-load torque, and the slip at full load torque is 0.028. Neglect stator resistance and rotational losses. Find the maximum torque in per unit of full load torque and the slip at which it takes place. Find the starting torque in per unit of full load torque.
Problem 6.10
The slip at full load for a three-phase induction motor is 0.04 and the rotor current at starting is 5 times its value at full load. Find the starting torque in per unit of full-load torque and the ratio of the maximum torque to full load torque and the slip at which it takes place.

Problem 6.11
A 220-V three phase four-pole 60 Hz squirrel-cage induction motor develops a maximum torque of 250 percent at a slip of 14 percent when operating at rated voltage and frequency. Now, assume that the motor is operated at 180 V and 50 Hz. Determine the maximum torque and the speed at which it takes place.

Problem 6.12
A six-pole, 60-Hz three-phase wound rotor induction motor has a rotor resistance of 0.8 \( \Omega \) and runs at 1150 rpm at a given load. The motor drives a constant torque load. Suppose that we need the motor to run at 950 rpm while driving the same load. Find the additional resistance required to be inserted in the rotor circuit to fulfil this requirement.

Problem 6.13
Assume for a 3-phase induction motor that for a certain operating condition the stator \( \bar{I} \bar{R} \) = rotor \( \bar{I} \bar{R} \) = core loss = rotational loss and that the output is 30 KW at 86% efficiency. Determine the slip under this operating condition.

Problem 6.14
Find the required additional rotor resistance to limit starting current to 45 A for a 3-phase 600-V induction motor with \( R_T = 1.66 \Omega \) and \( X_T = 4.1 \Omega \).

Problem 6.15
The rotor \( I_2 \bar{R} \) at starting are 6.25 times that at full load with slip of 0.035 for a three-phase induction motor. Find the slip at maximum torque and the ratio of starting to full-load torques.

Problem 6.16
The following parameters are available for a single-phase induction motor

\[
R_1 = 1.5 \Omega \quad R_2' = 3.4 \Omega \\
X_1 = X_2' = 3 \Omega \quad X_m = 100 \Omega
\]

Calculate \( Z_f \), \( Z_o \), and the input impedance of the motor for a slip of 0.06.

Problem 6.17
The induction motor of Problem 6.16 is a 60-Hz 110-V four-pole machine. Find the output power and torque under the conditions of Problem 6.16 assuming that the core losses are 66 W. Neglect rotational losses.
Problem 6.18
A four-pole 110-V 60-Hz single-phase induction motor has the following parameters:

\[ R_1 = 0.8 \, \Omega \quad R'_2 = 1 \, \Omega \]
\[ X_1 = X'_2 = 1.92 \, \Omega \quad X_m = 42 \, \Omega \]

The core losses are equal to the rotational losses, which are given by 40 W. Find the output power and efficiency at a slip of 0.05.

Problem 6.19
The following parameters are available for a single-phase 110-V induction motor:

\[ R_1 = R'_2 = 2.7 \, \Omega \]
\[ X_1 = X'_2 = 2.7 \, \Omega \]
\[ X_m = 72 \, \Omega \]

The core losses are 18.5 W and rotational losses are 17 W. Assume that the machine has four poles and operates on a 60-Hz supply. Find the rotor ohmic losses, output power, and torque for a slip of 5%.

Problem 6.20
The stator resistance of a single-phase induction motor is 1.96 \, \Omega and the rotor resistance referred to the stator is 3.6 \, \Omega. The motor takes a current of 4.2 A from the 110-V supply at a power factor of 0.624 when running at slip of 0.05. Assume that the core loss is 36 W and that the approximation of Eq. (6.47) is applicable. Find the motor’s output power and efficiency neglecting rotational losses.

Problem 6.21
A single-phase induction motor takes an input power of 280 W at a power factor of 0.6 lagging from a 110-V supply when running at a slip of 5 percent. Assume that the rotor resistance and reactance are 3.38 and 2.6 \, \Omega, respectively, and that the magnetizing reactance is 60 \, \Omega. Find the resistance and the reactance of the motor.

Problem 6.22
For the motor of Problem 6.21, assume that the core losses are 35 W and the rotational losses are 14 W. Find the output power and efficiency when running at a slip of 5 percent.

Problem 6.23
The output torque of a single-phase induction motor is 0.82 N \cdot m at a speed of 1710 rpm. The efficiency is 60 percent and the fixed losses are 37 W. Assume that motor operates on a 110-V supply and that the stator resistance is 2 \, \Omega. Find
the input power factor and input impedance. Assume that the rotor ohmic losses are 35.26 W. Find the forward and backward gap power and the values of $R_f$ and $R_b$. Assume a four-pole machine.

**Problem 6.24**
The forward field impedance of a $\frac{1}{2}$-hp four-pole 110-V 60-Hz single-phase induction motor for a slip of 0.05 is given by

$$Z_f = 12.4 + j16.98 \Omega$$

Assume that

$$X_m = 53.5 \Omega$$

Find the values of the rotor resistance and reactance.

**Problem 6.25**
For the motor of Problem 6.24, assume that the stator impedance is given by

$$Z_i = 1.86 + j2.56 \Omega$$

Find the internal mechanical power, output power, power factor, input power, developed torque, and efficiency, assuming that friction losses are 15 W.